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Max Flow Min Cut Property:
A Conjecture

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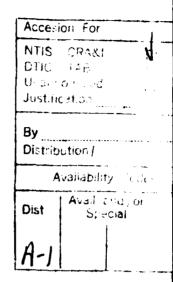
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A *clutter* C is a collection $E(C)$ of subsets of a finite set $V(C)$ with the property that $A_1 \not\subseteq A_2$ for all $A_1, A_2 \in E(C)$. Let $M(C)$ be the $0,1$ matrix with columns indexed by $V(C)$ whose rows are the incidence vectors of the members of $E(C)$.

Consider the two linear programs

$$min\{wx : \ x \geq 0, \ M(C)x \geq 1\} \qquad (1)$$

$$max\{y1 : \ y \geq 0, \ yM(C) \leq w\}. \qquad (2)$$

The clutter C has the *Max Flow Min Cut property* if (1) and (2) have integral optimum solutions x and y for all nonnegative integral vectors w. C is *ideal* if (1) has an integral optimum solution x for all nonnegative (integral) vectors w. C *packs* if (1) and (2) have integral optimum solutions x and y when $w = 1$.

For a clutter C, the *deletion* $C\backslash j$ and *contraction* C/j of an element $j \in V(C)$ are clutters defined as follows: $V(C\backslash j) = V(C/j) = V(C) - \{j\}$, $E(C\backslash j) = \{A \in E(C) : j \notin A\}$ and $E(C/j)$ are the minimal members of

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$\{A - \{j\} : A \in E(\mathcal{C})\}$. Any clutter obtained from $\mathcal{C}$ by repeated application of the contraction and deletion operations is called a *minor* of $\mathcal{C}$.

We propose the following conjecture.

Conjecture 1 *A clutter $\mathcal{C}$ has the Max Flow Min Cut property if and only if $\mathcal{C}$ and all its minors pack.*

Remark 1 The "only if" part of the above statement is obvious since saying that a minor of $\mathcal{C}$ packs is equivalent to stating that (1) and (2) have integral solution vectors x and y for the objective function $w_j = 0$ for a deleted element j, $w_j = \infty$ for a contracted element j and $w_j = 1$ for the remaining elements.

Remark 2 A weakening of the "if" condition follows from Lehman's characterization of ideal clutters [2] which implies the following:
If $\mathcal{C}$ is not ideal, it contains a minor $\mathcal{C}'$ such that the linear program (1) associated with $\mathcal{C}'$ has a unique optimum solution x whose components are all fractional when $w = 1$. It follows that if $\mathcal{C}$ and all its minors pack, then $\mathcal{C}$ is ideal.

Remark 3 A special case where the conjecture holds is when $\mathcal{C}$ is a binary clutter. Seymour [4] proved that a binary clutter has the Max Flow Min Cut property if and only if it does have a $\mathcal{Q}_6$ minor, where $V(\mathcal{Q}_6) = \{1, 2, 3, 4, 5, 6\}$ and $E(\mathcal{Q}_6) = \{\{1, 3, 5\}, \{1, 4, 6\}, \{2, 3, 6\}, \{2, 4, 5\}\}$. It is easy to check that the linear program (2) associated with $\mathcal{Q}_6$ has a unique optimum solution vector $y = \frac{1}{2}$ when $w = 1$.

Remark 4 This conjecture is the exact analog of the replication lemma used in the proof of the Fulkerson-Lovász [1] [3] pluperfect graph theorem. Let

$$max\{wx : x \geq 0, \ M(\mathcal{C})x \leq 1\} \tag{3}$$

$$min\{y1 : y \geq 0, \ yM(\mathcal{C}) \geq w\}. \tag{4}$$

The replication lemma shows that (3) and (4) have integral optimum solution vectors x and y for every $0, 1$ vector w if and only if they have integral optimum solution vectors for every (nonnegative) integral vector w.

The pluperfect graph theorem states that if (4) has an integral optimum solution y for every $0, 1$ vector w, then (3) and (4) have integral optimum solutions x and y for every (nonnegative) integral vector w.

Remark 5 To prove the "if" part of Conjecture 1, it is sufficient to show the following.

Conjecture 2 (Replication Conjecture) *If C and all its minors pack, then the clutter C_j defined below. packs. For $j \in V(C)$, let*

$$V(C_j) = V(C) \cup \{j'\}$$

$$E(C_j) = E(C) \cup (\bigcup_{A \ni j} A \setminus \{j\} \cup \{j'\}).$$

Indeed, observe that the linear programs (1) and (2) have integral optimum solutions for the vector w such that $w_j = 2$ and $w_i = 1$ for $i \neq j$ if and only if the clutter C_j packs. Therefore, using the replication conjecture recursively, it follows that C has the Max Flow Min Cut property.

References

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